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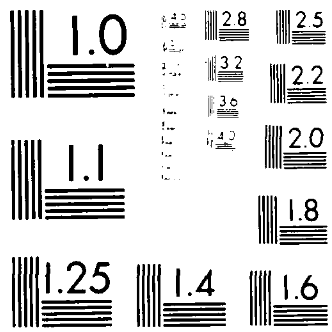

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Abstract. Suppose that $\{\xi_j\}$ is a strictly stationary sequence which satisfies the strong mixing condition. Denote by $M_n^{(k)}$ the k-th largest value of $\xi_1, \xi_2, \ldots, \xi_n$, and $\{u_n(\cdot)\}$ a sequence of normalizing functions for which $P[M_n^{(1)} \le u_n(x)]$ converges weakly to a continuous distribution $G(x)$. It is shown that if for some $k = 2, 3, \ldots$, $P[M_n^{(k)} \le u_n(x)]$ converges for each x, then there exist probabilities $p_1, \ldots, p_{k-1}$ such that

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19) Abstract (cont'd.)

$P[M_n^{(j)} \leq u_n(x)]$ converges weakly to $G(x)[1 + \sum_{i=1}^{j-1} \frac{(-\log G(x))^i}{i!} p_i]$ for $j = 2, \ldots, k$, where natural interpretations can be given for the p_j. This generalizes certain results due to Dziubdziela (J. Appl. Prob. 21, 720-729 (1984)), and Hsing et al. (Technical Report No. 150, Center for Stochastic Processes, UNC). It is further demonstrated that, with minor modification, the technique can be extended to study the joint limiting distribution of the order statistics. In particular, Theorem 1 of Welsch (Ann. Math. Statist. 43, 439-446 (1972)) is generalized, and some links between the convergence of the order statistics and that of certain point processes are established.

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ON THE EXTREME ORDER STATISTICS FOR A STATIONARY SEQUENCE

by

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Technical Report No. 198

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and
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Abstract. Suppose that $\{\xi_j\}$ is a strictly stationary sequence which satisfies the strong mixing condition. Denote by $M_n^{(k)}$ the k-th largest value of $\xi_1, \xi_2, \ldots, \xi_n$, and $\{u_n(\cdot)\}$ a sequence of normalizing functions for which $P[M_n^{(1)} \leq u_n(x)]$ converges weakly to a continuous distribution $G(x)$. It is shown that if for some $k = 2, 3, \ldots$, $P[M_n^{(k)} \leq u_n(x)]$ converges for each x, then there exist probabilities $p_1, \ldots, p_{k-1}$ such that $P[M_n^{(j)} \leq u_n(x)]$ converges weakly to $G(x)[1 + \sum_{i=1}^{j-1} \frac{(-\log G(x))^i}{i!} p_i]$ for $j = 2, \ldots, k$, where natural interpretations can be given for the p_j. This generalizes certain results due to Dziubdziela (J. Appl. Prob. 21, 720-729 (1984)), and Hsing et al. (Technical Report No. 150, Center for Stochastic Processes, UNC). It is further demonstrated that, with minor modification, the technique can be extended to study the joint limiting distribution of the order statistics. In particular, Theorem 1 of Welsch (Ann. Math. Statist. 43, 439-446 (1972)) is generalized, and some links between the convergence of the order statistics and that of certain point processes are established.

EXTREME VALUES POINT PROCESSES WEAK CONVERGENCE

1. Introduction

Let $\{\xi_j\}$ be a strictly stationary sequence of random variables satisfying the strong mixing condition (also known as uniform or α-mixing). For each n, let $M_n^{(1)} \geqq M_n^{(2)} \geqq \ldots \geqq M_n^{(n)}$ be the order statistics of $\xi_1, \ldots, \xi_n$, and write M_n for $M_n^{(1)}$ for convenience. Suppose there exist normalizing functions v_n, $n \geqq 1$, and a continuous type distribution function G for which $P[M_n \leqq v_n(x)] \xrightarrow{w} G(x)$, where $\xrightarrow{w}$ denotes weak convergence. The following questions can be asked:

(a) Does $P[M_n^{(k)} \leqq v_n(x)]$ converge weakly for each $k \geqq 2$?

(b) If, for some $k \geqq 2$, $P[M_n^{(k)} \leqq v_n(x)]$ converges weakly, how is the limit characterized?

In the i.i.d. setting the answers to the above questions are well known (cf. Leadbetter et al. (1983)); namely for each $k \geqq 2$,

$$P[M_n^{(k)} \leqq v_n(x)] \xrightarrow{w} G(x)\left(1 + \sum_{j=1}^{k-1} \frac{(-\log G(x))^j}{j!}\right)$$

where $0 \log 0 := 0$. For a dependent sequence, however, the answer to (a) is not necessarily affirmative. Mori(1976) provides an example of $\{\xi_j\}$ for which $P[M_n \leqq v_n(x)]$ converges weakly, but $P[M_n^{(2)} \leqq v_n(x)]$ does not. Exploiting the ideas in Mori(1976), it is possible to construct examples to show that for any fixed $k \geqq 2$, the weak convergence of $P[M_n^{(j)} \leqq v_n(x)]$, $1 \leqq j \leqq k-1$, does not in general garantee that of $P[M_n^{(k)} \leqq v_n(x)]$. However, the following question is unanswered:

(a') Suppose, for some $k \geqq 3$, $P[M_n^{(k)} \leqq v_n(x)]$ converges weakly. Does it follow that $P[M_n^{(j)} \leqq v_n(x)]$, $2 \leqq j \leqq k-1$, all converge weakly?

With regard to (b) in the dependent case, two papers are relevant.

Under certain constraints, Dziubdziela(1984) and Hsing et al.(1986) characterize the limiting distribution of $P[M_n^{(k)} \leq v_n(x)]$, assuming that $P[M_n^{(k)} \leq v_n(x)]$ converges weakly for each k. In view of the examples mentioned in the previous paragraph, their studies, though useful, are not sufficient to answer (b).

In this paper some problems connected with the above (a') and (b) are considered. First, in section 2, we briefly discuss the assumptions stated earlier, and prove a technical lemma. We then study in section 3, for any fixed k, the necessary and sufficient conditions for $P[M_n^{(k)} \leq v_n(x)]$ to have a limiting distribution. There answers to both (a') and (b) are obtained. It is seen in section 4 that the method in section 3 can be extended to study the limit of $P[M_n^{(1)} \leq v_n(x), M_n^{(k)} \leq v_n(y)]$ for any fixed k, and, in particular, a result in Welsch(1972) is generalized. Finally, in section 5, we discuss the connection of the convergence of the order statistics and that of certain point processes which were studied in Hsing(1985) and Hsing et al. (1986).

2. Preliminaries

To avoid repeated reference, assume without further mention that the conditions in first paragraph of section 1 hold throughout the paper. It is known that the strong mixing condition is often too stringent for the purpose of extremal theory. Nevertheless it is technically convenient, and to replace it by a more appropriate mixing condition is now considered straightforward (cf. Leadbetter et al.(1983), and Hsing et al.(1986)). That G is continuous is hardly a restriction; it is the case if, say, G is of extreme value type (cf. Leadbetter et al.(1983)). Under this assumption, there exist normalizing functions u_n for which

$$\lim_{n\to\infty} P[M_n \leq u_n(\tau)] = e^{-\tau}, \quad \tau > 0.$$

For notational convenience we shall throughout work exclusively with u_n.

For later reference, we state without proof the following lemma which is a version of some well-known results (cf. Loynes(1965) and Leadbetter et al.(1983)).

<u>Lemma 2.1</u> For each $\sigma > 0$ and $\tau > 0$,

$$\lim_{n\to\infty} P[M_{[\sigma n]} \leq u_n(\tau)] = \lim_{n\to\infty} P[M_n \leq u_{[\frac{n}{\sigma}]}(\tau)] = \lim_{n\to\infty} P[M_n \leq u_n(\sigma\tau)] = e^{-\sigma\tau},$$

where, here and hereafter, $[y]$ denotes the integer part of y. Thus it follows that if $\sigma_1 < \sigma_2$, $u_{[n/\sigma_1]}(\tau) > u_n(\sigma_2\tau)$ and $u_n(\sigma_1\tau) > u_n(\sigma_2\tau)$ for all sufficiently large n.

It is of interest to consider whether parallels of Lemma 2.1 exist for order statistics other than the maximum. The following lemma solves this problem.

<u>Lemma 2.2</u> Suppose for some $k \geq 2$, $\tau > 0$, and $\sigma_u > \sigma_\ell > 1$, either

$P[M^{(k)}_{[\sigma n]} \leq u_n(\tau)]$ or $P[M^{(k)}_n \leq u_n(\sigma\tau)]$ converges for each σ in (σ_ℓ, σ_u). The for each σ in (σ_ℓ, σ_u), $\lim_{n\to\infty} P[M^{(k)}_{[\sigma n]} \leq u_n(\tau)] = \lim_{n\to\infty} P[M^{(k)}_n \leq u_n(\sigma\tau)]$.

Proof First assume that $P[M^{(k)}_n \leq u_n(\sigma\tau)]$ converges for each σ in (σ_ℓ, σ_u). For σ and σ' with $\sigma_\ell < \sigma < \sigma' < \sigma_u$,

$$
\begin{aligned}
(2.1)\qquad \limsup_{n\to\infty} P[M^{(k)}_{[\sigma' n]} \leq u_n(\tau)] &= \limsup_{n\to\infty} P[M^{(k)}_{[\sigma'[n/\sigma']]} \leq u_{[n/\sigma']}(\tau)] \\
&= \limsup_{n\to\infty} P[M^{(k)}_n \leq u_{[n/\sigma']}(\tau)] \leq \lim_{n\to\infty} P[M^{(k)}_n \leq u_n(\sigma\tau)].
\end{aligned}
$$

Here the first equality follows from the identity $\{n: n \geq 1\} = \{[n/\sigma']: n \geq 1\}$, the second equality holds since $0 \leq n - [\sigma'[n/\sigma']] \leq \sigma'$ and $P[M_{[\sigma']} > u_{[n/\sigma']}(\tau)] \to 0$, and the inequality follows from Lemma 2.1. Similarly, for σ and σ'' with $\sigma_\ell < \sigma'' < \sigma < \sigma_u$,

$$
(2.2)\qquad \liminf_{n\to\infty} P[M^{(k)}_{[\sigma'' n]}(k) \leq u_n(\tau)] \geq \lim_{n\to\infty} P[M^{(k)}_n \leq u_n(\sigma\tau)].
$$

By (2.1) and (2.2), for σ and σ_i, $1 \leq i \leq 4$, with $\sigma_\ell < \sigma_1 < \sigma_2 < \sigma < \sigma_3 < \sigma_4 < \sigma_u$,

$$
\begin{aligned}
\limsup_{n\to\infty} P[M^{(k)}_{[\sigma_4 n]} \leq u_n(\tau)] &\leq \lim_{n\to\infty} P[M^{(k)}_n \leq u_n(\sigma_3\tau)] \leq \lim_{n\to\infty} P[M^{(k)}_n \leq u_n(\sigma\tau)] \\
&\leq \lim_{n\to\infty} P[M^{(k)}_n \leq u_n(\sigma_2\tau)] \leq \liminf_{n\to\infty} P[M^{(k)}_{[\sigma_1 n]} \leq u_n(\tau)].
\end{aligned}
$$

But

$$
\begin{aligned}
\liminf_{n\to\infty} P[M^{(k)}_{[\sigma_1 n]} \leq u_n(\tau)] &- \limsup_{n\to\infty} P[M^{(k)}_{[\sigma_4 n]} \leq u_n(\tau)] \\
&\leq \limsup_{n\to\infty} (P[M^{(k)}_{[\sigma_1 n]} \leq u_n(\tau)] - P[M^{(k)}_{[\sigma_4 n]} \leq u_n(\tau)]) \\
&\leq \lim_{n\to\infty} P[M_{[(\sigma_4-\sigma_1)n]} > u_n(\tau)] = 1 - e^{-(\sigma_4-\sigma_1)\tau}
\end{aligned}
$$

which tends to zero if $\sigma_4 - \sigma_1 \to 0$. This shows that $\lim_{n\to\infty} P[M^{(k)}_n \leq u_n(\cdot\tau)]$ is continuous at σ. Since for σ, σ_1, and σ_2 with $\sigma_\ell < \sigma_1 < \sigma < \sigma_2 < \sigma_u$,

$$\lim_{n\to\infty} P[M_n^{(k)} \leq u_n(\sigma_1\tau)] \leq \liminf_{n\to\infty} P[M_{[\sigma n]}^{(k)} \leq u_n(\tau)] \leq \limsup_{n\to\infty} P[M_{[\sigma n]}^{(k)} \leq u_n(\tau)]$$
$$\leq \lim_{n\to\infty} P[M_n^{(k)} \leq u_n(\sigma_2\tau)]$$

by (2.1) and (2.2), it is easily seen that $P[M_{[\sigma n]}^{(k)} \leq u_n(\tau)]$ converges and has the same limit as does $P[M_n^{(k)} \leq u_n(\sigma\tau)]$.

Suppose now $P[M_{[\sigma n]}^{(k)} \leq u_n(\tau)]$ converges for each σ in (σ_ℓ, σ_u). Using arguments similar to the ones in getting (2.1) and (2.2), it can be seen that for σ, σ_1, and σ_2 with $\sigma_\ell < \sigma_1 < \sigma < \sigma_2 < \sigma_u$,

$$\lim_{n\to\infty} P[M_{[\sigma_2 n]}^{(k)} \leq u_n(\tau)] \leq \liminf_{n\to\infty} P[M_n^{(k)} \leq u_n(\sigma\tau)] \leq \limsup_{n\to\infty} P[M_n^{(k)} \leq u_n(\sigma\tau)]$$
$$\leq \lim_{n\to\infty} P[M_{[\sigma_1 n]}^{(k)} \leq u_n(\tau)].$$

As before, the difference between $\lim_{n\to\infty} P[M_{[\sigma_1 n]}^{(k)} \leq u_n(\tau)]$ and $\lim_{n\to\infty} P[M_{[\sigma_2 n]}^{(k)} \leq u_n(\tau)]$ tends to zero as σ_1 and σ_2 tend to σ. This concludes the proof.

Q.E.D.

We remark that, by applying the triangle inequality, Lemma 2.2 can be extended to situations where finitely many order statistics are involved. In particular, Lemma 2.2 remains true if, in the statement of the lemma, $P[M_n^{(k)} \leq u_n(\sigma\tau)]$ and $P[M_{[\sigma n]}^{(k)} \leq u_n(\tau)]$ are replaced by $P[M_n^{(1)} \leq u_n(\sigma\tau),\ M_n^{(k)} \leq u_n(\sigma\tau')]$ and $P[M_{[\sigma n]}^{(1)} \leq u_n(\tau),\ M_{[\sigma n]}^{(k)} \leq u_n(\tau')]$, respectively. This fact will be applied in section 4.

3. The Limiting Distribution of $M_n^{(k)}$

The essence of our theory lies in the fact that the sequence $\xi_1, \xi_2, \ldots$ can be divided into "asymptotically independent" groups $(\xi_{(i-1)r_n+1}, \ldots, \xi_{ir_n})$, $i \geq 1$, of size r_n each (in the precise sense as described by Lemma 3.1 below), where $\{r_n\}$ is determined in the following manner. Let $\{\ell_n\}$ be any sequence such that $\ell_n/n \to 0$ and $\alpha(\ell_n) \to 0$, where $\alpha(\cdot)$ is the mixing function of the strong mixing condition which holds for $\{\xi_j\}$, and let $\{r_n\}$ be such that

$$(3.1) \qquad n/r_n \to \infty, \quad e^{n/r_n}\,\alpha(\ell_n) \to 0, \quad \text{and} \quad e^{n/r_n}\,\ell_n/n \to 0.$$

For any such $\{\ell_n\}$ and $\{r_n\}$, it is not difficult to show (cf. Hsing et al. (1986, Lemma 2.2 and 2.3) that for each $\tau > 0$,

$$(3.2) \qquad \lim_{n\to\infty} e^{n/r_n}\, P[M_{\ell_n} > u_n(\tau)] = 0$$

and

$$(3.3) \qquad \lim_{n\to\infty} n/r_n\, P[M_{r_n} > u_n(\tau)] = \tau.$$

It will soon be clear that $\{\ell_n\}$ and $\{r_n\}$ only function as step stones in the proofs, and indeed the theory is independent of the specific choice of these sequences. The following lemma is essential.

<u>Lemma 3.1</u> Let $\tau > 0$, $\sigma > 0$, and $k = 2,3,\ldots$ be constants. Write $k_n = [\sigma n/r_n]$, and let $\hat{X}_{n,m}$, $1 \leq m \leq k_n$, be i.i.d. r.v.'s having the same distribution as does $\sum_{j=1}^{r_n} 1(\xi_j > u_n(\tau))$ where $1(\cdot)$ is the indicator function. Then

$$P[M_{[\sigma n]}^{(k)} \leq u_n(\tau)] - P[\sum_{m=1}^{k_n} \hat{X}_{n,m} \leq k-1] \to 0 \quad \text{as} \quad n \to \infty.$$

<u>Proof</u> Write $X_{n,m} = \sum_{j=(m-1)r_n+1}^{mr_n} 1(\xi_j > u_n(\tau))$, $1 \leq m \leq k_n$. Since

$P[M^{(k)}_{[\sigma n]} \leq u_n(\tau)] = P[\sum_{j=1}^{[\sigma n]} 1(\xi_j > u_n(\tau)) \leq k-1]$, it is easily shown that

$$(3.4) \qquad P[M^{(k)}_{[\sigma n]} \leq u_n(\tau)] - P[\sum_{m=1}^{k_n} X_{n,m} \leq k-1] \to 0.$$

For each fixed $s = 1,2,\ldots,$ the set $[\sum_{m=1}^{k_n} X_{n,m} = s]$ can be written as the union of $\binom{k_n+s-1}{s} = \frac{(k_n+s-1)!}{s!\,(k_n-1)!}$ disjoint sets of the form $[X_{n,m} = s_m,$ $1 \leq m \leq k_n]$ where $\sum_{1}^{k_n} s_m = s$. For each fixed choice of such s_m, $1 \leq m \leq k_n$,

$$|P[X_{n,m} = s_m,\ 1 \leq m \leq k_n] - \prod_{m=1}^{k_n} P[X_{n,m} = s_m]|$$
$$\leq (k_n - 1)\,(\alpha(\ell_n) + 2\,P[M_{\ell_n} > u_n(\tau)])$$

by some standard arguments (cf. Leadbetter et al.(1983)). Thus

$$(3.5) \qquad \begin{aligned} &|P[\sum_{m=1}^{k_n} X_{n,m} = s] - P[\sum_{m=1}^{k_n} \hat{X}_{n,m} = s]| \\ &\leq (k_n - 1)\binom{k_n+s-1}{s}(\alpha(\ell_n) + 2\,P[M_{\ell_n} > u_n(\tau)]. \end{aligned}$$

It is obvious that $(k_n - 1)\binom{k_n+s-1}{s} < e^{k_n}$ for large n. Thus the dominant side of (3.5) tends to zero by (3.1) and (3.2). The result follows on combining this with (3.4). Q. E. D.

For $i \geq 1$, write

$$(3.6) \qquad \pi_n(i;\tau) = P[\sum_{j=1}^{r_n} 1(\xi_j > u_n(\tau)) = i \mid \sum_{j=1}^{r_n} 1(\xi_j > u_n(\tau)) > 0]$$

and denote by $\pi_n^{*\ell}(\cdot;\tau)$ the ℓ-fold convolution of $\pi_n(\cdot;\tau)$, namely

$$(3.7) \qquad \pi_n^{*\ell}(i;\tau) = \begin{cases} 0, & i < \ell, \\ \displaystyle\sum_{\substack{i_1+\cdots+i_\ell=i \\ i_r \geq 1,\ 1\leq r\leq \ell}} \cdots \sum \pi_n(i_1;\tau) \cdots \pi_n(i_\ell;\tau), & i \geq \ell. \end{cases}$$

Corollary 3.2 $P[M_{[\sigma n]}^{(k)} \leq u_n(\tau)] = e^{-\sigma\tau} [1 + \sum_{\ell=1}^{k-1} \frac{(\sigma\tau)^\ell}{\ell!} \sum_{i=\ell}^{k-1} \pi_n^{*\ell} (i;\tau)] + o(1)$

where $\pi_n^{*\ell}(i;\tau)$ is defined by (3.7).

Proof $\sum_{m=1}^{k_n} 1(\hat{X}_{n,m} > 0)$ is distributed as binomial with mean $k_n P[\hat{X}_{n,1} > 0] = k_n P[M_{r_n} > u_n(\tau)]$ which tends to $\sigma\tau$ as n tends to ∞ by (3.3). Thus $\sum_{m=1}^{k_n} 1(\hat{X}_{n,m} > 0)$ converges in distribution to a Poisson variable with mean $\sigma\tau$, from which the result easily follows. Q.E.D.

The main result of this section is the following.

Theorem 3.3 Let $k \geq 2$ be a constant. If $P[M_n^{(k)} \leq u_n(\tau)]$ converges for each $\tau > 0$, then for any $\tau > 0$ and $1 \leq i \leq k-1$ the probability $\pi_n(i;\tau)$ defined in (3.6) converges to some $\pi(i)$ which is independent of τ, and, in this case,

$$(3.8) \qquad \lim_{n\to\infty} P[M_{[\sigma n]}^{(j)} \leq u_n(\tau)] = e^{-\sigma\tau} [1 + \sum_{\ell=1}^{j-1} \frac{(\sigma\tau)^\ell}{\ell!} \sum_{i=\ell}^{j-1} \pi^{*\ell}(i)],$$

$$\sigma > 0, \quad \tau > 0, \quad 2 \leq j \leq k,$$

where

$$\pi^{*\ell}(i) = \begin{cases} 0, & i < \ell, \\ \displaystyle\sum_{\substack{i_1+\dots+i_\ell=i \\ i_r\geq 1,\ 1\leq r\leq\ell}} \cdots \sum \pi(i_1) \cdots \pi(i_\ell), & i \geq \ell \end{cases}$$

Conversely if for some $\tau > 0$, $\pi_n(i;\tau)$ converges for $1 \leq i \leq k-1$, then $P[M_n^{(k)} \leq u_n(\tau)]$ converges for each $\tau > 0$.

Proof First assume that $P[M_n^{(k)} \leq u_n(\tau)]$ converges for each $\tau > 0$. Fix a $\tau > 0$ for now. By Lemma 2.2 and Corollary 3.2, $\sum_{\ell=1}^{k-1} \frac{(\sigma\tau)^\ell}{\ell!} \sum_{i=\ell}^{k-1} \pi_n^{*\ell}(i;\tau)$ converges for each $\sigma > 1$. This implies that $\sum_{i=\ell}^{k-1} \pi_n^{*\ell}(i;\tau)$, $1 \leq \ell \leq k-1$,

all converge. Thus $\sum_{i=k-1}^{k-1} \pi_n^{*k-1}(i;\tau) = [\pi_n(1;\tau)]^{k-1}$ converges, or $\pi_n(1;\tau)$ converges, and $\sum_{i=k-2}^{k-1} \pi_n^{*k-2}(i;\tau) = [\pi_n(1;\tau)]^{k-2} + (k-2)[\pi_n(1;\tau)]^{k-3}\pi_n(2;\tau)$ converges, which implies that $\pi_n(2;\tau)$ converges, etc. It follows from a simple induction that for each $1 \leq i \leq k-1$, $\pi_n(i;\tau)$ converges, say, to $\pi(i;\tau)$. Hence Corollary 3.2 implies

$$(3.9) \qquad \lim_{n\to\infty} P[M_{[\sigma n]}^{(j)} \leq u_n(\tau)] = e^{-\sigma\tau}\left[1 + \sum_{\ell=1}^{j-1} \frac{(\sigma\tau)^\ell}{\ell!} \sum_{i=\ell}^{j-1} \pi^{*\ell}(i;\tau)\right],$$

$$\sigma,\ \tau > 0,\ 2 \leq j \leq k.$$

To show (3.8), it now remains to show that $\pi(i;\tau)$ is independent of τ. Fix $\tau_2 > \tau_1$. It follows from (3.9) that for $2 \leq j \leq k$,

$$\lim_{n\to\infty} P[M_{[\frac{\tau_2}{\tau_1}n]}^{(j)} \leq u_n(\tau_1)] = e^{-\tau_2}\left[1 + \sum_{\ell=1}^{j-1} \frac{\tau_2^\ell}{\ell!} \sum_{i=\ell}^{j-1} \pi^{*\ell}(i;\tau_1)\right],$$

$$\lim_{n\to\infty} P[M_n^{(j)} \leq u_n(\tau_2)] = e^{-\tau_2}\left[1 + \sum_{\ell=1}^{j-1} \frac{\tau_2^\ell}{\ell!} \sum_{i=\ell}^{j-1} \pi^{*\ell}(i;\tau_2)\right].$$

But Lemma 2.2 implies that the two limits are the same for each $2 \leq j \leq k$, which in turn implies that $\pi(i;\tau_1) = \pi(i;\tau_2)$, $1 \leq i \leq k-1$. This proves (3.8)

It is worth noting that, in the above derivation, the assumption that $P[M_n^{(k)} \leq u_n(\tau)]$ converges for each $\tau > 0$ can be relaxed considerably; for example, it was enough to assume that $P[M_n^{(k)} \leq u_n(\tau)]$ converges for all $\tau \leq$ some $\tau_0 > 0$. We shall make use of this fact in the following part of the proof.

Conversely, suppose for some $\tau > 0$, $\pi_n(i;\tau)$ converges for $1 \leq i \leq k-1$. Then by Corollary 3.2, $P[M_{[\sigma n]}^{(k)} \leq u_n(\tau)]$ converges for each $\sigma > 0$. It thus

follows from Lemma 2.2 that $P[M_n^{(k)} \leq u_n(\sigma\tau)]$ converges for each $\sigma > 1$. The first part of the proof and the remark in the preceding paragraph now imply that $P[M_n^{(k)} \leq u_n(\tau)]$ converges for each $\tau > 0$. This concludes the proof. Q.E.D.

The following corollary is easily shown.

<u>Corollary 3.4</u> If for some $\tau > 0$, $\pi_n(1;\tau) \to 1$ as $n \to \infty$, then for all $k \geqq 1$ and $\tau > 0$,

$$\lim_{n\to\infty} P[M_n^{(k)} \leq u_n(\tau)] = e^{-\tau} \sum_{\ell=0}^{k-1} \frac{\tau^\ell}{\ell!} . \tag{3.10}$$

Conversely, if (3.10) holds for some $k \geqq 2$ and $\tau > 0$, then $\pi_n(1;\tau) \to 1$ as $n \to \infty$, and hence (3.10) holds for all $k \geqq 1$ and $\tau > 0$.

<u>Proof</u> Assume first that $\pi_n(1;\tau) \to 1$ as $n \to \infty$ for some $\tau > 0$. Then it is simply seen that $\pi_n(i;\tau) \to 0$ for all $i \geqq 2$, and (3.10) follows readily from the theorem. Next suppose (3.1) holds for some $k \geqq 2$ and $\tau > 0$. It follows from Corollary 3.2 that $\lim_{n\to\infty} \sum_{i=\ell}^{k-1} \pi_n^{*\ell}(i;\tau) = 1$ for all $\ell = 1,\dots,k-1$, which implies that $\lim_{n\to\infty} \pi_n(1;\tau) = 1$ and the conclusion follows from the first part. Q.E.D.

Note that the condition "$\pi_n(1;\tau) \to 1$" in Corollary 3.4 is reminiscent of the condition (17) in Loynes(1965), and the condition $D'(u_n)$ in Leadbetter(1974).

4. The Joint Limiting Distribution of $M_n^{(1)}$ and $M_n^{(k)}$

We now consider the normalized limits of $M_n^{(1)}$ and $M_n^{(k)}$ jointly for any fixed $k \geq 2$. In spirit of (3.6) and (3.7), define, for $\tau > \tau' > 0$,

$$\rho_n(i;\tau,\tau') = P[\sum_{j=1}^{r_n} 1(\xi_j > u_n(\tau')) = 0, \sum_{j=1}^{r_n} 1(\xi_j > u_n(\tau)) = i \mid \sum_{j=1}^{r_n} 1(\xi_j > u_n(\tau)) > 0],$$

(4.1)

$$\rho_n^{*\ell}(i;\tau,\tau') = \begin{cases} 0, & i < \ell, \\ \sum\limits_{\substack{i_1+\cdots+i_\ell=i \\ i_r \geq 1,\ 1\leq r\leq \ell}} \cdots \sum \rho_n(i_1;\tau,\tau') \cdots \rho_n(i_\ell;\tau,\tau'), & i \geq \ell, \end{cases}$$

where $\{r_n\}$ is obtained in (3.1). The following result parallels Theorem 3.3.

<u>Theorem 4.1</u> Let $k \geq 2$ be a constant. If $P[M_n^{(1)} \leq u_n(\tau'), M_n^{(k)} \leq u_n(\tau)]$ converges for each τ and $\tau' > 0$, then for any $\tau > \tau' > 0$ and $1 \leq i \leq k-1$, $\rho_n(i;\tau,\tau')$ converges to some $\rho(i;\tau'/\tau)$ which depends on τ and τ' through their ratio, and in this case for $\sigma,\tau,\tau' > 0$ and $2 \leq j \leq k$,

(4.2)

$$\lim_{n\to\infty} P[M_{[\sigma n]}^{(1)} \leq u_n(\tau'), M_{[\sigma n]}^{(j)} \leq u_n(\tau)] = \begin{cases} e^{-\sigma\tau'}, & 0 < \tau \leq \tau', \\ e^{-\sigma\tau}[1 + \sum\limits_{\ell=1}^{j-1} \frac{(\sigma\tau)^\ell}{\ell!} \sum\limits_{i=\ell}^{j-1} \rho^{*\ell}(i;\tau'/\tau)], & 0 < \tau' < \tau, \end{cases}$$

where

$$\rho^{*\ell}(i;s) = \begin{cases} 0, & i < \ell, \\ \sum\limits_{\substack{i_1+\cdots+i_\ell=i \\ i_r \geq 1,\ 1\leq r\leq \ell}} \cdots \sum \rho(i_1;s) \cdots \rho(i_\ell;s), & i \geq \ell. \end{cases}$$

Conversely, if there exists a τ such that $\rho_n(i;\tau,s\tau)$ converges for each

$0 < s < 1$ and $1 \leq i \leq k-1$, then $P[M_n^{(1)} \leq u_n(\tau'), M_n^{(k)} \leq u_n(\tau)]$ converges for each τ and $\tau' > 0$.

<u>Proof</u> We remarked after proving Lemma 2.2 that the result may be generalized to where two or more order statistics are involved. The same remark applies to Corollary 3.2, which can be extended to give

$$P[M_{[\sigma n]}^{(1)} \leq u_n(\tau'), M_{[\sigma n]}^{(j)} \leq u_n(\tau)] \tag{4.3}$$
$$= \begin{cases} e^{-\sigma\tau'} + o(1), & 0 < \tau \leq \tau', \\ e^{-\sigma\tau}\left[1 + \sum_{\ell=1}^{j-1} \frac{(\sigma\tau)^\ell}{\ell!} \sum_{i=\ell}^{j-1} \rho_n^{*\ell}(i;\tau,\tau')\right] + o(1), & 0 < \tau' < \tau, \end{cases}$$

for each $\sigma > 0$ and $j \geq 2$, where $\rho_n^{*\ell}$ is defined in (4.1). Suppose $P[M_n^{(1)} \leq u_n(\tau'), M_n^{(k)} \leq u_n(\tau)]$ converges for each $\tau, \tau' > 0$. It can be shown, as in the proof of Theorem 3.3, that for any $\tau > \tau'$ and $1 \leq i \leq k-1$, $\rho_n(i;\tau,\tau')$ converges to some $\rho(i;\tau,\tau')$ and it follows from (4.3) that

$$\lim_{n\to\infty} P[M_{[\sigma n]}^{(1)} \leq u_n(\tau'), M_{[\sigma n]}^{(j)} \leq u_n(\tau)] \tag{4.4}$$
$$= \begin{cases} e^{-\sigma\tau'}, & 0 < \tau \leq \tau, \\ e^{-\sigma\tau}\left[1 + \sum_{\ell=1}^{j-i} \frac{(\sigma\tau)^\ell}{\ell!} \sum_{i=\ell}^{j-1} \rho^{*\ell}(i;\tau,\tau')\right], & 0 < \tau' < \tau, \end{cases}$$

for any $\sigma > 0$ and $2 \leq j \leq k$. Take τ_1, τ_1', τ_2, and τ_2' such that $\tau_2 / \tau_1 = \tau_2' / \tau_1' > 1$ and $\tau_1' / \tau_1 = \tau_2' / \tau_2 < 1$. Then (4.4) implies that for $2 \leq j \leq k$,

$$\lim_{n\to\infty} P[M_{[\frac{\tau_2}{\tau_1} n]}^{(1)} \leq u_n(\tau_1'), M_{[\frac{\tau_2}{\tau_1} n]}^{(j)} \leq u_n(\tau_1)]$$
$$= e^{-\tau_2}\left[1 + \sum_{\ell=1}^{j-1} \frac{\tau_2^\ell}{\ell!} \sum_{i=\ell}^{j-1} \rho^{*\ell}(i;\tau_1,\tau_1')\right]$$

and

$$\lim_{n\to\infty} P[M_n^{(1)} \le u_n(\tau_2'),\ M_n^{(j)} \le u_n(\tau_2)]$$
$$= e^{-\tau_2}\left[1 + \sum_{\ell=1}^{j-1} \frac{\tau_2^{\ell}}{\ell!} \sum_{i=\ell}^{j-1} \rho^{*\ell}(i;\tau_2,\tau_2')\right].$$

But, since $\frac{\tau_2}{\tau_1}\tau_1' = \tau_2'$ and $\frac{\tau_2}{\tau_1}\tau = \tau_2$, it follows from the variant of Lemma 2.2 mentioned immediately after Lemma 2.2 that the two limits are the same for $2 \le j \le k$. This shows that $\rho(i;\tau,\tau')$, $1 \le i \le k-1$, depend on τ and τ' through τ/τ' and (4.2) is proved. The remaining steps of this proof parallel those in the proof of Theorem 3.3 and are therefore left for the reader. Q.E.D.

It can be observed from the above proof and the proof of Theorem 3.3 that our method lends itself to still more general situations. In fact, the limiting form of $P[M_n^{(k_i)} \le u_n(\tau_i),\ 1 \le i \le I]$ can be thus determined for any fixed choice of $k_1, k_2, \ldots, k_I$, and I. However, we shall spare the details since not much more content can be added by making them specific.

Some properties of the probability $\rho(i;s)$ in Theorem 4.1 are included in the following result.

<u>Theorem 4.2</u> Let $k \ge 2$ be fixed. Assume that $P[M_n^{(1)} \le u_n(\tau'),\ M_n^{(k)} \le u_n(\tau)]$ converges for each τ and $\tau' > 0$. Then the probabilities $\rho(i;s)$, $0 < s < 1$, $1 \le i \le k-1$, in Theorem 5.1 satisfy the following properties:

(a) $\rho(i;s)$ is nonincreasing in s,

(b) $0 \le \sum_{\ell=1}^{k-1} \rho(\ell;s) \le 1-s$ for each $s \in (0,1)$,

(c) $\lim_{s\to 0} \rho(i;s) = \pi(i) := \lim_{n\to\infty} P[\sum_{j=1}^{r_n} 1(\xi_j > u_n(\tau)) = i \mid \sum_{j=1}^{r_n} 1(\xi_j > u_n(\tau)) > 0]$,

(d) $\sum_{\ell=1}^{i} \rho(\ell;s)$, as a function of s, is concave,

where (a), (c), and (d) hold for each $i = 1,\ldots, k-1$.

Proof By (3.3) and Theorem 4.1,

$$\rho(i;s) = \lim_{n\to\infty} \frac{n}{r_n} P[\sum_{j=1}^{r_r} 1(\xi_j > u_n(s)) = 0, \sum_{j=1}^{r_n} 1(\xi_j > u_n(1)) = i].$$

That (a) holds is trivial. To show (b), observe that

$$0 \leq \sum_{\ell=1}^{k-1} \rho(\ell;s) = \lim_{n\to\infty} \frac{n}{r_n} P[\sum_{j=1}^{r_n} 1(\xi_j > u_n(s)) = 0, \sum_{j=1}^{r_n} 1(\xi_j > u_n(1)) \leq k-1]$$

$$\leq \lim_{n\to\infty} \frac{n}{r_n} P[\sum_{j=1}^{r_n} 1(\xi_j > u_n(s)) = 0, \sum_{j=1}^{r_n} 1(\xi_j > u_n(1)) > 0]$$

$$= \lim_{n\to\infty} \frac{n}{r_n} (P[\sum_{j=1}^{r_n} 1(\xi_j > u_n(1)) > 0] - P[\sum_{j=1}^{r_n} 1(\xi_j > u_n(s)) > 0])$$

$$= \lim_{n\to\infty} \frac{n}{r_n} (\frac{r_n}{n} - \frac{r_n s}{n}) = 1 - s$$

by (3.3), and this shows (b). It can be shown similarly that

$$0 \leq \frac{n}{r_n} (P[\sum_{j=1}^{r_n} 1(\xi_j > u_n(1)) = i] - P[\sum_{j=1}^{r_n} 1(\xi_j > u_n(s)) = 0,$$

$$\sum_{j=1}^{r_n} 1(\xi_j > u_n(1)) = i])$$

$$\leq \frac{n}{r_n} P[\sum_{j=1}^{r_n} 1(\xi_j > u_n(s)) > 0] \xrightarrow[n\to\infty]{} s \xrightarrow[s\to 0]{} 0.$$

Thus $P[\sum_{j=1}^{r_n} 1(\xi_j > u_n(1)) = i | \sum_{j=1}^{r_n} 1(\xi_j > u_n(1)) > 0]$ converges if $\rho(i;s)$ converges as $s \to 0$, which it dose since $\rho(i;s)$ is bounded above by one and is nonincreasing. This proves (c). It remains to show (d). For this we write $\rho(s) = \sum_{\ell=1}^{i} \rho(\ell;s)$ for a fixed i, and follow the steps in Theorem 1 of Welsch(1973). It suffices to show that for each $0 < r < s < 1$ and $\varepsilon > 0$ for which $s + \varepsilon s < 1$, we have

$$(4.5)\qquad \frac{\rho(s+\varepsilon s)-\rho(s)}{\varepsilon s} \leq \frac{\rho(r+\varepsilon r)-\rho(r)}{\varepsilon r}.$$

For each selection of such r, s, and ε, we can find $0 < \tau_1' < \tau_2' < \tau_1 < \tau_2 < 1$ by letting

$$\tau_1' < r,\quad \tau_2' = \tau_1' + \varepsilon\tau_1',\quad \tau_1 = \tau_1' / s,\quad \tau_2 = \tau_1' / r.$$

Thus $s = \tau_1' / \tau_1$, $r = \tau_1' / \tau_2$, $\varepsilon s = \frac{\tau_2'-\tau_1'}{\tau_1'} \cdot \frac{\tau_1'}{\tau_1} = \frac{\tau_2'-\tau_1'}{\tau_1}$, and $\varepsilon r = \frac{\tau_2'-\tau_1'}{\tau_1'} \cdot \frac{\tau_1'}{\tau_2'} = \frac{\tau_2'-\tau_1'}{\tau_2'}$. In terms of the τ's, (4.5) becomes

$$(4.6)\qquad \tau_1 \left[\rho\left(\frac{\tau_2'}{\tau_1}\right) - \rho\left(\frac{\tau_1'}{\tau_1}\right)\right] \leq \tau_2 \left[\rho\left(\frac{\tau_2'}{\tau_2}\right) - \rho\left(\frac{\tau_1'}{\tau_2}\right)\right],$$

which we now show. It is readily seen from (3.3) that for $\tau < \tau'$

$$\tau\, \rho\left(\frac{\tau'}{\tau}\right) = \lim_{n\to\infty} \frac{n}{r_n} P\left[\sum_{j=1}^{r_n} 1\,(\xi_j > u_n(\tau')) = 0,\ \sum_{j=1}^{r_n} 1\,(\xi_j > u_n(\tau)) \leq i\right]$$

$$(4.7)\qquad = \lim_{n\to\infty} \frac{n}{r_n} P[M_{r_n}^{(1)} \leq u_n(\tau'),\ M_{r_n}^{(i+1)} \leq u_n(\tau)].$$

Since for all large n

$$P[M_{r_n}^{(1)} \leq u_n(\tau_2'),\ M_{r_n}^{(i+1)} \leq u_n(\tau_1)] - P[M_{r_n}^{(1)} \leq u_n(\tau_1'),\ M_{r_n}^{(i+1)} \leq u_n(\tau_1)]$$

$$\leq P[M_{r_n}^{(1)} \leq u_n(\tau_2'),\ M_{r_n}^{(i+1)} \leq u_n(\tau_2)] - P[M_{r_n}^{(1)} \leq u_n(\tau_1'),\ M_{r_n}^{(i+1)} \leq u_n(\tau_2)],$$

(4.6) follows simply from (4.7). This concludes the proof. Q.E.D.

Welsch(1972) proved the claims in Theorem 4.1 and Theorem 4.2 (a), (b), and (d) for the case $k = 2$, assuming that there are constants a_n, b_n, and a distribution function G such that $\lim_{n\to\infty} P[M_n \leq a_n x + b_n] = G(x)$. In this connection, Mori(1976) showed that (a), (b), and (d) of Theorem 4.2 fully

characterize the cluster probability $\rho(1;s)$ in the sense that for each function $\rho(s)$ satisfying the three conditions, one can construct an α-mixing stationary sequence $\{\xi_j\}$ for which there exist constants a_n, b_n, and a distribution function G such that

$$\lim_{n\to\infty} P[M_n^{(1)} \leqq a_n x + b_n,\ M_n^{(2)} \leqq a_n y + b_n\}$$

$$= \begin{cases} G(x), & y \geqq x, \\ G(y)\{1 - \rho[(\log G(x) / \log G(y)] \log G(y)\}, & y < x. \end{cases}$$

5. The Convergence of Certain Point Processes

For notation and theory of point processes we follow Kallenberg(1983). Hsing et al.(1986) studied the so-called exceedance point process $N_n^{(\tau)}$ which consists of the points $\{j/n: \xi_j > u_n(\tau), 1 \leq j \leq n\}$. It was shown there that if $N_n^{(\tau)}$ converge in distribution w. r. t. the vague topology in the space of locally finite counting measures on $(0, 1]$, the limit must be compound Poisson. The following result states the connection between the convergence of $N_n^{(\tau)}$ and that of $P[M_n^{(k)} \leq u_n(\tau)]$.

Theorem 5.1 $N_n^{(\tau)}$ converges in distribution for each $\tau > 0$ w. r. t. the vague topology in the space of locally finite counting measures on $(0, 1]$ if and only if for each $\tau > 0$, $P[M_n^{(k)} \leq u_n(\tau)]$ converges for each $k \geq 1$, and

$$\lim_{k\to\infty} \lim_{n\to\infty} P[M_n^{(k)} \leq u_n(\tau)] = 1 \tag{5.1}$$

Proof If $N_n^{(\tau)}$ converges in distribution to $N^{(\tau)}$, then by the continuous mapping theorem $P[M_n^{(k)} \leq u_n(\tau)] = P[N_n^{(\tau)} (0, 1] \leq k-1]$ converges to $P[N^{(\tau)} (0, 1] \leq k-1]$ as n tends to ∞, and hence

$$\lim_{k\to\infty} \lim_{n\to\infty} P[M_n^{(k)} \leq u_n(\tau)] = \lim_{k\to\infty} P[N^{(\tau)} (0, 1] \leq k-1] = 0.$$

Suppose next that the converse is true. Then $\pi_n(i;\tau) \to$ some $\pi_n(i)$ for each i, and

$$\begin{aligned} 1 &= \lim_{k\to\infty} \lim_{n\to\infty} P[M_n^{(k)} \leq u_n(\tau)] \\ &= \lim_{k\to\infty} e^{-\tau} \left[1 + \sum_{\ell=1}^{k-1} \frac{\tau^\ell}{\ell!} \sum_{i=\ell}^{k-1} \pi^{*\ell}(i)\right] \end{aligned} \tag{5.2}$$

$$= e^{-\tau} \left[1 + \sum_{\ell=1}^{\infty} \frac{\tau^{\ell}}{\ell!} \sum_{i=\ell}^{\infty} \pi^{*\ell}(i)\right]$$

by virtue of Theorem 3.3 and monotone convergence. But (5.2) implies that $\sum_{i=\ell}^{\infty} \pi^{*\ell}(i) = 1$ for each ℓ, or, equivalently, $\sum_{i=1}^{\infty} \pi(i) = 1$. That $N_n^{(\tau)}$ converges in distribution follows from Theorem 4.2 of Hsing et al.(1986).

Q.E.D.

In addition to $\lim_{n\to\infty} P[M_n \leq u_n(\tau)] = e^{-\tau}$, $\tau > 0$, we now require that, for each n, u_n be nonincreasing, left continuous, and such that

$$\lim_{\substack{\tau_1\to 0 \\ \tau_2\to\infty}} P[u_n(\tau_2) < \xi_1 < u_n(\tau_1)] = 1.$$

Define $u_n^{-1}(\xi) = \sup\{\tau > 0 : \xi \leq u_n(\tau)\}$. $u_n^{-1}(\xi) < \tau$ if and only if $\xi > u_n(\tau)$. Consider the two-dimensional point process N_n which has the points $\{(j / n,\ u_n^{-1}(\xi_j)) : j \geq 1\}$. The limiting distributions of point processes of this type were studied in Pickands (1971), Resnick (1975), Weissman (1974), Mori (1977), and Hsing (1985). The following result was obtained by Hsing (1985), in which a detailed proof can be found.

<u>Theorem 5.2</u> If N_n converges in distribution to N w. r. t. the vague topology in the space of locally finite counting measures on $\mathbb{R}_+ \times \mathbb{R}_+ = (0, \infty) \times (0, \infty)$, then N consists of the points $\{(S_i, T_i Y_{ij}): i \geq 1, 1 \leq j \leq K_i\}$ where (S_i, T_i), $i \geq 1$, are the points of a mean one Poisson process [illegible] on $\mathbb{R}_+ \times \mathbb{R}_+$, Y_{ij}, $1 \leq j \leq K_i$, are the points of a point process γ_i on $[1, \infty)$ with 1 as an atom, $\gamma_1, \gamma_2, \ldots$ are identically distributed, and [illegible], $\gamma_1, \gamma_2, \ldots$ are mutually independent.

Sketch of Proof It can be shown that a point process ζ has the representation described in the theorem if and only if it satisfies the following properties:

(i) $\zeta \circ g_{\alpha,\beta} \overset{d}{=} \zeta$ for each $\alpha, \beta > 0$, where $g_{\alpha,\beta}(x, y) := (\alpha x + \beta, \alpha^{-1} y)$, $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$;

(ii) For any choice $I_1, \ldots, I_k$ of disjoint intervals of the form $[a, b)$ in $\mathbb{R}_+$, and any choice $J_1, \ldots, J_m$ of intervals of the form $[c, d)$ in $\mathbb{R}_+$, the m-dimensional random vectors $(\zeta(I_i \times J_1), \ldots, \zeta(I_i \times J_m))$, $1 \leq i \leq k$, are mutually independent, where k and m are arbitrary positive integers;

(iii) $P[\zeta((0, 1) \times (0, \tau)) > 0] = e^{-\tau}$, $\tau > 0$.

For the point process N in the present theorem, (i) follows from stationarity of $\{\xi_i\}$ and a variant of Lemma 2.2, (ii) holds since $\{\xi_j\}$ is α-mixing, and (iii) follows from the assumption that $\lim_{n\to\infty} P[M_n \leq u_n(\tau)] = e^{-\tau}$, $\tau > 0$.

Q.E.D.

The interpretation of the convergence of N_n, in terms of the order statistics, can be summarized to give the following result which we state without proof.

Theorem 5.3 N_n converges in distribution w. r. t. the vague topology in the space of locally finite counting measures on $\mathbb{R}_+ \times \mathbb{R}_+$ if and only if $P[M_n^{(k_i)} \leq u_n(\tau_i), 1 \leq i \leq I]$ converges for each choice of $\tau_i > 0$, $k_i \geq 1$, $1 \leq i \leq I$, $I \geq 1$, and (5.1) holds for each $\tau > 0$.

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